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ORIGINAL ARTICLE

The ostensive dimension through the lenses of two didactic approaches

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Abstract The paper presents how two different theories—the APC-space and the ATD—can frame in a complementary way the semiotic (or ostensive) dimension of mathematical activity in the way they approach teaching and learning phenomena. The two perspectives coincide in the same subject: the importance given to ostensive objects (gestures, discourses, written symbols, etc.) not only as signs but also as essential tools of mathematical practices. On the one hand, APC-space starts from a general semiotic analysis in terms of "semiotic bundles" that is to be integrated into a more specific epistemological analysis of mathematical activity. On the other hand, ATD proposes a general model of mathematical knowledge and practice in terms of "praxeologies" that has to include a more specific analysis of the role of ostensive objects in the development of mathematical activities in the classroom. The articulation of both theoretical perspectives is proposed as a contribution to the development of suitable frames for Networking Theories in mathematics education.

1 Introduction

It is well known that there is a strong contrast in mathematical activities between the abstract nature of mathematical objects, which are usually seen as having no perceptual existence, and their representations, which are tangible and upon which subjects' activities can develop in a concrete way. The management of such a duality is basic in all learning processes. It is important therefore to develop suitable frameworks to analyse this duality and to clarify its role in the teaching and learning of mathematics. These frameworks cannot avoid facing the epistemological question of the nature and "substance" of the objects involved in the mathematical activity.

The mentioned duality has been afforded by Bosch and Chevallard (1999) within the *Anthropological Theory of the Didactic* (ATD), introducing the dialectic between what they call "*ostensives*" and "*non-ostensives*". They observed that there is a variety of palpable registers, or ostensives, through which mathematical activities can develop:

> "...the *oral* register, the *trace* register (which includes all graphic stuff and writing products), the *gesture* register, and lastly the register of what we can call the *generic materiality*, for lack of a better word, namely the register where those ostensive objects that do not belong to any of the registers above reside" (Bosch & Chevallard, 1999, p. 96, emphasis in the original, translation from the French by the authors).

On the one hand, mathematical activity cannot go on without making use of sets of "ostensive objects" belonging to these palpable registers. On the other hand, doing mathematics cannot be reduced to dealing with ostensives: a crucial element is the dialectic between these "palpable" objects and the "non-ostensives" which are evocated, represented or "embodied" by the ostensives, and the practices within which they are treated.

As Bosch and Chevallard (1999) point out, ostensives have a twofold function: they have a "semiotic value",

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linked to their power to stand for non-ostensives objects, and an "instrumental value", namely their function as tools of mathematical practices.

The semiotic function of ostensives in mathematics classroom has been studied by Arzarello (Arzarello, 2006; Arzarello & Olivero, 2005) through a broadened notion of semiotic system (the so called *Semiotic Bundle*), and framed within the so called *Space of Action–Production–Communication* (APC-space) model.

According to the aims of this Special Issue, the major goal of the paper consists in starting a comparison of the two frames and in proposing a complementary use of the two theories for approaching *didactic problems* in which the ostensive dimension appear to be of special relevance.

The next section of the paper sketchily exposes how the two frames approach ostensives. The subsequent section contains the main result of the paper, namely a (tentative) analysis of the ostensives through the "complementary" use of the two frames. The last section elaborates some comments on the added value gained by combining the two theories, within the frame of Networking Theories.

2 The ATD and APC-space approaches

Looking at the phenomenology of learning processes in the mathematics classes, a variety of *ostensives* are observable. They may be produced or used with great flexibility: the same subject generally exploits simultaneously more than one of them (e.g. speech and gesture; formulas and graphs, etc.). Sometimes these ostensive resources are shared by the students (and possibly by the teacher) and used as communication tools, other times they can have a more private status and reveal as crucial thinking tools. All such (ostensive) resources, with the actions and productions they enhance, appear important in the building of mathematical ideas and the development of mathematical activities. In fact they reveal crucial to bridge the gap between the timeless and context-less sentences of formal mathematics and the everyday experience through which we make sense of concepts, included mathematical ones (non-ostensive objects). These general observations suggest that in order to scientifically describe the learning processes in the classroom, it is necessary to consider all such *resources*, how they *evolve*, and the *practices* they are treated with.

It is also important to notice that not all such resources have the same status in the classroom. In fact, the instrumental and semiotic value that is attributed to ostensives can deeply vary according to the nature of the ostensive itself, to the context, and to the subjects' judgement. Words, graphs and written formalisms are more seen as signs than as tools of the activity, while gestures and others material objects are rarely considered as relevant tools of the mathematical activity. On the other side, written formalisms often happen to result less meaningful and with less evocative power for the students.

In the next subsections we expose sketchily the theoretical frames of the ATD and of the APC-space, underlining their specific approaches to ostensives.

2.1 The ATD and the ostensive dimension of didactic phenomena

From the ATD point of view, the ostensive dimension of mathematical activity is interesting as far as it makes it possible to shed light on the mathematics educational problems. It thus has to be integrated in a model of a globally considered mathematical activity, which includes all kind of practices leading to the construction, development, utilisation and diffusion of mathematics (Chevallard, 1999, 2002, 2004, 2006; Bosch & Gascón, 2006). The ATD assumes an institutional conception of the mathematical activity and proposes to model mathematical notions and practices in terms of "praxeologies":

> "Mathematics, like any other human activity, is something that is produced, taught, learned, practiced and diffused in social institutions. It can be modelled in terms of praxeologies called mathematical praxeologies or mathematical organizations" (García, Gascón, Ruiz Higueras, & Bosch 2006, p. 226).

A praxeology (or mathematical organization) is structured in two levels: the *praxis* or "know how", which includes different kinds of problems to be studied as well as techniques available to solve them and the *logos* or "knowledge", which includes the theoretical discourses that describe, explain and justify the techniques used. The "knowledge" block includes two progressive levels of justification: the "technology" (as a first discourse—"logos"—about the "technè") and the "theory" (as a second order justification or a "technology of the technology"). An example of mathematical praxeology is given by García et al. about the "proportion problems":

> "a set of problematic tasks (the classic proportional problems where three measures are given and a fourth one is to be found), techniques to deal with these problems (commonly known as *rule of three*) and a technological–theoretical discourse that explains and justifies the mathematical activity performed (defining what are proportional magnitudes and how to determine if two magnitudes are directly or inversely proportional)" (García et al., 2006).

Mathematical praxeologies are the object of learning and teaching in the schools and are almost always related to a given institution:

> "The mathematical knowledge is produced, taught, learned, practised and diffused in social institutions. It is thus not possible to separate it from its process of construction in a specific institution" (García et al., 2006).

The ATD includes the ostensive dimension of mathematical activity in the core of its epistemological model, that is, as the "flesh" in which praxeologies are embodied. Thus, any praxeology is always activated through the manipulation of ostensives and the evocation of non-ostensives (e.g. the ostensive $y = k{\cdot}x$, the expression "linear function" or the graph of a straight line and the evocation of the abstract mathematical concept "linear function"), which are like the two sides of the same coin. In general, our culture tends to overvalue the non-ostensives (the concepts), while the ostensives are underestimated, especially gestures, words and informal graphisms. They are valued according to their semiotic function, that is they are considered as signs or perceivable objects whose only function is to represent other objects. But ATD points out another important, usually neglected function of ostensives: the instrumental function. In fact, ostensives are not simple working media but genuine instruments for the mathematical activity: their careful manipulation not only allows performing a mathematical task but is also essential for its accomplishment, e.g. for solving an equation. The instrumental and the semiotic values of the ostensive objects depend on the practices of the institutional system, where they are activated. Consequently the non-ostensive objects exist because of the manipulation of the ostensive ones within specific praxeological organisations.

The mathematical world is composed by both ostensive and non-ostensive objects. Concepts are as essential for the mathematical activity to be developed, as symbols, words and gestures are; mathematical praxeologies (and, more generally, any human praxeology) is made of ostensives and non-ostensives. What is more, all ingredients of praxeologies (type of tasks, techniques, technologies and theories) are made of ostensives and non-ostensives. The study of the mathematical praxeologies has to take into account the specific role played by the ostensives. It may happen that a mathematical problem is not stated, a technique does not evolve, or a justification is not made because a concept is missing but also because a word, a graph or a symbol is not available. Given that the ATD formulates a didactic phenomenon in terms of the creation, development, evolution and diffusion of praxeologies, the ostensive dimension becomes an important component of any problem, question, fact or experience that can be studied.

Some of the didactic problems that have been recently studied in the ATD approach refer to phenomena which have a relevant ostensive dimension. We can mention, for instance, those related to the constraints that hinder the normal existence of mathematical modelling activities at school (García et al., 2006). These constraints are particularly patent in the case of the teaching of algebra as a modelling tool at lower secondary schools (Bolea, Bosch & Gascón, 2004), in the difficulties to teach activities related to algebraic-functional modelling at upper secondary schools (Ruiz, Bosch & Gascón, 2008) or even at university level (Barquero, Bosch & Gascón, 2008). In short, the general problem is to give a clear mathematical status to most of the ostensive tools that are required during the modelling process and that do not belong to the official mathematical world: verbal expressions, gestures, informal graphs, etc. The prevailing tendency to reduce the process of building and composing mathematical models to the use of "prefabricate" and ready-to-use symbolic and written models does not make the situation easier and contributes to the reduction of the necessary "ostensive thickness" of the mathematical modelling practice.

2.2 Embodiment, multimodality and APC-space

The APC-space approach is particularly apt to focus the dynamics among the different ostensives used by the students in the short time-scales of the classroom story. It allows to study classroom events that take place in few minutes or even seconds and that are considered crucial for the evolution of knowledge, namely the *actions* made by pupils, their *productions* (in different languages: verbal, gestural, written), the *communicative* interactions (between students, and between students and teacher). Hence the name of *Space of Action, Production and Communication* (in short, APC-space).

The approach is based on a fundamental hypothesis on the formation of concepts in learners, and more in general on knowledge formation, that is called *multimodality*, within the paradigm of *embodiment*, which has been developed in the last decade (for a synthetic overview, see Wilson, 2002). Embodiment is a stream afoot in cognitive science that grants the body a central role in shaping the mind. It concerns different disciplines, e.g. cognitive science and neuroscience, interested in how the body is involved in thinking and learning. The new stance emphasizes sensory and motor functions, as well as their importance for successful interaction with the environment. A major consequence is that the boundaries among perception, action and cognition become *porous* (Seitz, 2000). Concepts are so analysed not on the basis of "formal abstract models, totally unrelated to the life of the body, and of the brain regions governing the body's functioning in the world" (Gallese & Lakoff, 2005, p. 455), but considering the *multimodality* of our cognitive performances. Verbal language itself (e.g. metaphorical productions) is "part of these cognitive multimodal activities" (Gallese & Lakoff, 2005). In the more extreme version, the frame of multimodality appears to suggest that "the understanding of a mathematical concept rather than having a definitional essence, spans diverse perceptuomotor activities, which become more or less active depending of the context" (Nemirovsky, 2003, p. 108).

The APC-space frames mathematical learning processes according to the multimodal paradigm. Namely it makes it possible to consider how action and perception determine the processes of learning and to describe them so that doing, touching, moving and seeing appear as their important multimodal ingredients. Specifically, the APC-space is meant to be a model for framing the processes that develop and are possibly shared in the classroom among students (and the teacher) while working together (Arzarello, 2008; Arzarello & Olivero, 2005). It analyses them considering their different components and a variety of mutually dependent relationships among them. The components are the body, the physical world, and the cultural environment: in a word, the students themselves and the teacher along with the context where they are acting and learning. When students learn mathematics, these and other components (e.g. the emotional ones) take an active part in the learning processes, interacting together. The interaction comes from the students' work, the teacher's mediation and possibly from the use of artefacts. The three letters A, P, C illustrate the main dynamic relationships among such components, i.e. students' actions and interactions (e.g. in a situation at stake, with their mates, with the teacher, with themselves, with tools), their productions (e.g. answering a question, posing other questions, making a conjecture, introducing a sign to represent a situation, and so on) and communication aspects (e.g. when the discovered solution is communicated to a mate or to the teacher orally or in written form, using suitable representations). The APC-space is a typical complex system, which cannot be described in a linear manner as resulting by a simple superposition of its ingredients. It particularly models how the relationships among its components develop in the classroom through the specific actions of the teacher. The APC-space analysis intends also to account for how the multimodal aspects of learning processes come to be related to cultural and institutional aspects. In fact, as pointed out by L. Radford:

> "an account of the embodied nature of thinking must come to terms with the problem of the relationship between the body as a locus for the constitution of an individual's subjective meanings and the historically constituted cultural system of meanings and concepts that exists prior to that particular individual's actions" (Radford et al., 2005).

Ostensives can be framed suitably through the APC-space frame: we see them as constituting a palpable aspect of multimodality. To focus their nature and mutual relationships it is useful to use a semiotics lens, which is an excellent tool to enter into APC-space and its multimodal complexity. However, the classical semiotic approaches (for an updated survey see Sáenz-Ludlow & Presmeg, 2006) put strong limitations upon the structure of the semiotic systems they consider and therefore in our view they reveal too narrow to describe the complexity of APC-space ingredients, particularly of the ostensives. This happens for two reasons: (1) Students and teachers use a variety of semiotic resources in the classroom: words (orally or in written form); extra-linguistic modes of expression (gestures, glances, actions, etc.); different types of inscriptions (drawings, sketches, graphs, etc.); instruments (from the pencil to the most sophisticated ICT devices), and so on. Analysing such resources, we find that some of them do not satisfy the requirements of the classical definitions for semiotic systems as discussed in the literature (e.g. see Ernest, 2006; Duval, 2006). (2) The way in which such different resources are activated is multimodal, as pointed out above.

To overcome such limitations, Arzarello (2006) has introduced a broader semiotic tool: the *Semiotic Bundle* (Arzarello, 2006), a tool suitable to analyse the variety of resources and their relationships within the APC-space frame. Encompassing all the classical semiotic systems or registers as particular cases, the Semiotic Bundle does not contradict the semiotic analysis developed using such tools but broaden it with the double aim of getting new results and framing the old ones within a unitary wider picture. To define the Semiotic Bundle, we first need the notion of *Semiotic Set*, which broadens that of semiotic system.[1] A Semiotic Set is:

(a) A set of signs which may possibly be produced with different actions that have an intentional character, such as uttering, speaking, writing, drawing, gesticulating, handling an artefact, and so on.
(b) A set of modes for producing such signs and possibly transforming them; these modes can possibly be rules or algorithms but can also be more flexible action or production modes used by the subject (e.g. in gesturing, in drawing, etc.).
(c) A set of relationships among these signs and their meanings, e.g. between the sign " = " and its

[1] The definition of Semiotic Set is a generalisation of the definition of Semiotic System, as it is given in Ernest (2006, pp. 69–70).

meanings, or between a gesture and its meaning (e.g. see the classification in Goldin-Meadow, 2003, p. 6: iconic, metaphoric, deictic, beat gestures).

Examples of semiotic sets include all the usual semiotic systems (speech, written languages, the algebraic register, etc.), but also "new entries" like, gestures, drawings, sketches, etc. In fact, the three components above (signs, modes of production/transformation and relationships) may characterize a variety of resources, spanning from the compositional systems, usually studied in traditional semiotics (e.g. formal languages), to the open sets of signs (e.g. sketches, drawings, gestures). The former are made of elementary constituents and their rules of production involve both atomic (single) and molecular (compound) signs. The latter have holistic features, cannot be split into atomic components, and their modes of production and transformation are often idiosyncratic to the subject, who produces them. The word "set" must be interpreted in a very wide sense, e.g. as a variable collection.

Now we can define a *Semiotic Bundle* as the couple formed by:

A collection of semiotic sets.
A set of relationships between the sets of the bundle.

A semiotic bundle is not to be considered as a juxtaposition of semiotic sets; on the contrary, it is a unitary system and it is only for the sake of analysis that we distinguish its components as semiotic sets. A semiotic bundle is a *dynamic structure*, which changes in time because of the semiotic activities of the subject: for example, the collection of semiotic sets that constitute it may change; as well, the relationships between its components may vary in time.

Semiotic bundles can provide the semiotic lenses through which one can observe the nature and the dynamics of the ostensives in the APC-space. An interesting example is constituted by the couple speech-gesture. Recent research on gestures has uncovered some important relationships between the two (e.g. see Goldin-Meadow, 2003). There is strong evidence that gestures are so closely linked with speech that "we should regard the gesture and the spoken utterance as different sides of a single underlying mental process" (McNeill, 1992, p.1), namely "gesture and language are one system" (McNeill, 1992, p.2). In the APC-space frame, gesture and speech form a semiotic bundle, made of two deeply intertwined semiotic sets (of which only one, speech, is also a semiotic system).[2]

Arzarello and his team have used the APC-space frame and the Semiotic Bundle tool to study several phenomena that happen in teaching–learning context in the classroom (Arzarello et al., 2006; Arzarello, 2006). The emphasis is on the psychological and social aspects in learning processes. Besides to the multimodality of the studied phenomena, the analysis is particularly attentive to the cultural aspects that they reveal. In particular, the cultural aspects are present in the mediating action of the teacher. In this sense, the APC-space frame allows to embrace both the psychological and the cultural dimension of learning. Specifically, it can give reason for both the biological and the cultural roots of the ostensives produced and acted on in the classroom.

3 The ostensive dimension of didactic phenomena: a comparison combining ATD and APC-space

Both the ATD and APC-space frames focus on the ostensives as a relevant part of mathematics learning in a dialectic relationship with the non-ostensives. Hence both approaches do not tackle the learning of mathematics as a pure learning of concepts. This point is supported in complementary ways through an institutional analysis in the ATD theory and in a cognitive analysis in the APC-space frame. In this section we shall see how both frames can contribute to a more complete analysis of the ostensives.

Didactical phenomena can be analysed according to different time and space scales, which span from the small-scale flying moment of a learning process in a specific classroom as described in the APC-space to the long term and wide events, which produce the praxeologies at regional level described in the ATD. To grasp properly didactical phenomena, we argue it is fruitful to integrate theoretical frameworks based on complementary scales of analysis. Certainly coordinating the fine grain analysis of short-term processes with the analysis of long term processes is a difficult problem. On the one hand, APC-space allows to develop a fine-grained cognitive analysis (where the semiotic facts are interpreted within the APC components, which heavily refer to subjects' actions, productions and communications); on the other hand, the ATD frame makes it possible to develop an analysis from a cultural and institutional point of view (praxeology with techniques, technologies, theories, didactic transposition etc.). What happens in the classroom concerns both dimensions and it is our claim that an analysis carried out by combining the two approaches can benefit from their complementarity and can therefore give us rich information and interesting interpretations.

3.1 An example: chirographic reduction versus genetic conversion

We shall illustrate our tentative of combining the two frames by applying it to two related didactical and

[2] Another example, made of gazes, speech, gestures and inscriptions has been studied by F. Ferrara in her PhD Dissertation (Ferrara, 2006).

cognitive phenomena concerning ostensives, namely the *chirographic reduction*, studied in the ATD frame, and the *genetic conversion*, analysed in the APC-space frame. The notion of *chirographic reduction* (in Greek "$\chi\varepsilon\iota\rho$" means "hand") has been studied in Bosch and Chevallard (1999): they point out the "individual micro genesis of techniques for solving specific problems" (p. 104), that is a process that starting from ostensive objects (in discursive, gesture, graphic, written form) ends with stable techniques, generally developed on the sheet of paper. The chirographic reduction consists exactly in the "transfer of gestures and material objects to the oral and graphic registers, that is, those than can be easily transferred to the sheet of paper" (Bosch & Chevallard, 1999, p. 105). For example they analyse gesture and speech, which accompany the accomplishment of matrix product. In the end, these ostensive objects are integrated in new mathematical objects, represented through the algebraic formalism, where each trace of gesture and oral activity is eliminated. This is at the root of a didactical paradox. On the one hand, the genuine mathematical job seems to consist in "pure computation and pure syntax" and, in this sense, chirographic reduction can be considered as a necessary step in the evolution of mathematical activity, being it personal or institutional. The other ostensive aspects, which are embedded in the stream of time, do not seem to acquire a clear mathematical status. On the other hand, it is exactly the combination of this private component with the official one—i.e. the semiotic sets simultaneously active in a semiotic bundle— that seems able to give meaning to the official mathematical formalism.

A simple didactic consequence of this reduction is the difficulty to give a specific treatment to the ostensives that are not yet "chirographied", that is the material objects, gestures, informal graphs or oral expressions that act as real tools of the mathematical activity but finally disappear from the formal presentations of the results. Their role as components of the techniques can thus be forgotten in favour of the benefit of the concepts, meanings or ideas that are supposed to lead the development of the activity. The reduction of the mathematical ostensive "thickness" to what can be transferred to a sheet of paper can contribute to the "rigidity" of the mathematical practices done by the students: the activity is reduced to the traces left on the paper.

The ostensive reduction that affects mathematical practices is especially visible when analysing the process of mathematical modelling as it is defined by the ATD in terms of links between praxeologies (see García, 2005; García et al., 2006). It must be pointed out that the chirographic reduction in itself cannot explain (and even less provoke) the constraints that hinder mathematical modelling practices at school. Chirographic reduction can only be considered as an important facet of the *semiotic dimension* of this phenomenon. As such, it may appear sometimes as more visible and can help us detect the factors that make modelling practices difficult to be developed in school. It can help to precise the scope of the phenomenon and indicate what kind of elements have to be taken into account. In other words, instead of considering the chirographic reduction as the "cause", "source" or "explanation" of the constraints affecting the teaching of mathematics as a modelling process, our proposal is to study the relationships with other observable facts or phenomena that are deeply connected to these constraints.

An important fact related to the impoverishment of mathematical modelling practices and closely linked to the chirographic reduction is the concrete way of interpreting (or thinking about) mathematical activity in the considered institution. For instance, in those institutions where the euclideanism is the dominant epistemological model,[3] mathematical activity tends to be reduced to the sequence "definition–speculation–theorem–proof", that is, what Thurston (1994) defines as the "popular model" of mathematics. This particular vision of mathematics introduces a gap between mathematical crystallized knowledge and the process of construction, diffusion and utilisation of this knowledge. It increases the distance between (a) what is considered as "mathematical", that is, what can be made explicit through the written-symbolic register, can be used and controlled by well-defined rules, is thus easy to evaluate and usually put under the students' responsibility; and (b) what is considered as "didactic", related to all that is done to teach mathematics and is not considered as properly mathematical in nature, that uses all kind of registers (words, gestures, etc.) not considered as mathematical and which is under the teacher's responsibility. This division is at the root of the phenomenon of the students' "mathematical irresponsibility" (Chevallard, Bosch & Gascón, 1997) because it impedes the evolution of the didactic contract in the sense that students started progressively sharing the responsibility to manage some important dimensions of the mathematical activity (Rodríguez, 2005; Rodríguez et al., 2004, 2008).

In short, the chirographic reduction and, more especially, its relation with the material empowerishment of mathematical activity has been interpreted taking into account both the epistemological model of mathematics that is currently accepted in teaching institutions and the

[3] The "euclideanism", as a general epistemological model of mathematics, pretends to reduce mathematical activity to the process of deducing theorems from a finite set of propositions (the *axioms*) concerning the so called *primitive terms*, which are only indirectly defined. The truth of the axioms flows from the axioms to the theorems through deductive canals of truth transmission (the *proofs*) (Lakatos, 1978).

Fig. 1 A student makes use of gestures to represent in iconic way the context of a given problem

Fig. 2 Iconic written inscriptions that freeze the gestures

distribution of responsibilities assigned by the didactic contract which are related to the way and grade of integration (or separation) between what is considered as "mathematics" and what is considered as "didactics" in the corresponding teaching organisation.

Within the APC-space frame, the chirographic reduction can be analyzed through the notion of semiotic bundle, which puts forward how this process corresponds to some crucial *genetic conversion* from a semiotic set to another within a bundle, which dynamically develops. The genetic conversion is similar but not identical to the notion of conversion studied by Duval (2006). An example is illustrated in Figs. 1–3, and is extensively analyzed in Arzarello et al. (2006). Some pupils use gestures to model an arithmetic problem (Fig. 1), then the gestures are transformed in written inscriptions that freeze their gestures into an icon written on the paper (Fig. 2); subsequently the pupils operate on the inscriptions to model properly the situation and in the end activate an arithmetic system to interpret it (Fig. 3).

From a semiotic point of view, the initial gestures are generating the written signs on the paper. We call this a genetic conversion from the semiotic set of gestures to that of drawings, and later to the arithmetical semiotic system. In fact, the initial gestures of Fig. 1 have a genetic function with respect to the written signs of Figs. 2 and 3.

In another example (Arzarello, 2006), students describe a function that they must produce starting from a numerical table, first through gestures and words and only later through a graph or a formula. In this case, they produce a genetic conversion from the semiotic set of gestures to the semiotic system of Cartesian graphs. This genetic aspect of the process is not encompassed in the standard notion of conversion between registers (Duval, 2006), which presupposes conversions to act between two already existing semiotic systems, e.g. from the sign "$y = x^2$" to the graph of a curve. In the genetic conversion, on the contrary, there is a genesis of signs from a semiotic set to a fresh semiotic set or system. The signs introduced in the new set (system) are often built preserving some features of the previous signs, namely have iconic features (e.g. like the icon of a house preserves some of the features of a house, according to some cultural stereotype). The preservation generally concerns some of the extra-linguistic features of the previous signs, which are generating new signs within the new semiotic set (or system). Usually the genesis continues with successive (genetic or standard) conversions from the new sets (systems) into already codified systems, as observed in the previous analysis of ostensives through the ATD lens.

The chirographic reduction has been studied within the ATD frame from an institutional point of view; using the APC-space language, such reduction can be described as the tendency in mathematics to convert some semiotic set (e.g. gestures) into some other semiotic system (e.g. the arithmetic language). These systems can be represented with written inscriptions that can be treated through precise algorithms within a more rigid semiotic system, as observed in the above analysis of ostensives through the ATD lens (as an example, think to the conversion from the abacus praxeologies to the techniques of arithmetic). The APC-space analysis shows that, while doing that, the students are embedded in the stream of culture because of the multimodal way of their learning processes: gestures and idiosyncratic inscriptions are deeply blended with arithmetic within an evolving semiotic bundle. Sometimes this conversion develops spontaneously, sometimes it does not; in these cases the semiotic mediation of the teacher is crucial. A further observation coming from the analysis by means of the APC-space analysis is that the conversion (reduction) does not mean a definitive pruning of the old semiotic sets (ostensives): in fact, the semiotic set from which the genetic conversion has been generated can become active later (or produce new stable ostensives). For example, when the students tackle some difficult task the old semiotic set may appear again in the semiotic bundle and can be useful to support their processes. In this sense, the symbols so produced or used within the semiotic bundle often maintain an indexical function with respect to the older semiotic set [we use the dialectic index vs.

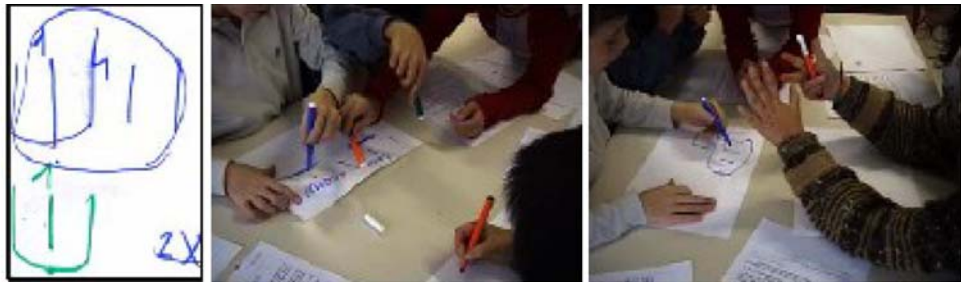

Fig. 3 Students work on the inscriptions and activate an arithmetic system to model the situation

symbol like in Peirce: see Arzarello (2006), or Sabena (2007) for more details]. More than a pruning, the genetic conversion is a flexible enlargement of the semiotic bundle, which can vary in time according to the didactic situations tackled by the students but often contains more or less explicitly the old semiotic sets that have generated the more formal new ones. Sometimes all these resources of the bundle are active and interacting very closely each other.

4 Discussion

It is clear that any mathematical activity is carried out by means of the manipulation of ostensive objects: writings, discourses, gestures and other material objects (including electronic and "virtual" ones, visually perceptible). The ostensive dimension characterizes mathematics in their true essence. In particular, this is true for the activity of *studying and helping others studying mathematics*, what has been named the *didactic activity* (Chevallard et al., 1997), which includes the teaching and learning at school as well as the construction of "new" mathematics (in a given institution), and the utilisation of mathematics to solve any kind of problem: all these activities can be described in terms of a sequence of ostensives and transformations between them.

Anyway, the omnipresence of the ostensive dimension in any didactic phenomenon should not lead us to think that the object of study of the didactics of mathematics can be reduced to this dimension or formulated in exclusively semiotic terms. In fact, any human activity has a semiotic dimension as it has sociological, psychological, cognitive and physiological ones. It is not our contempt to homogenize the emerging phenomena, reducing them to their semiotic dimension. Rather, semiotics as a science can take the mathematical activity and its teaching and learning as an object of study and raise interesting problems about it. What is more, elucidating these problems and the results obtained can be very useful for didactic researchers as far as it is properly interpreted within a didactic approach. Of course, it may lead to a *modification of the concepts imported from semiotics* as it happened with the concepts derived from epistemology, psychology, sociology or even mathematics (Brousseau, 1997). This has been done both in ATD and in APC-space approaches.

The ATD places semiotic analysis in relation to its own theoretical categories to avoid assuming without any control the implicit assumptions carried out by this analysis, especially those related to the cultural interpretation of mathematical activities. It thus tries to put forward "didactic varieties" of the semiotic notions when it seems useful to study didactic phenomena.

It can be said that the ATD felt the need to take into account the semiotic dimension of didactic phenomena after having elaborated its own modelling of mathematical and didactic institutionalised activities. Bosch and Chevallard (1999); Bosch (1994) place the notion of ostensive object and the dialectic between the ostensive and the non-ostensive as basic ingredients of praxeologies: praxeologies and their four components (types of problems, techniques, technologies and theories) are made of complexes of ostensives and non-ostensives. They highlight in what sense mathematical praxeologies are "sensitive" to the ostensives they are composed of; that is, how the availability (or non-availability) of a word, a written symbol or a gesture can affect the evolution of a praxeology and its development in the hands of a person or in a given institution.

The problem that can be raised is how to describe these complexes of ostensives and their function in the institutional praxeological dynamics. For instance, if we consider the case of proportionality and the linear function mentioned at the beginning of the paper, in order to understand the conditions of evolution of the mathematical activity carried out by the students in a class, it is important to know what ostensive tools the students are allowed to use and how these ostensives can help (or hinder) the relation between proportionality and other taught mathematical praxeologies. We can think, for instance, about the writing of a table of numbers, of an equality of rations or a functional notation. What words are associated to these writings: "quantities", "variables", "extremes" and "means", "numerator" or "denominator"? What gestures are being embodied to the specific treatment of such a relation: a "cross", a "vertical" link between quantities, a "horizontal" connection between values, etc.?

Furthermore, it can be of great importance to analyse what kind of ostensives are considered as an integrant part of the praxeology (and, consequently, are given a special and explicit treatment in the class) and what are relegated to a mere representation of the knowledge, thus appearing as of a second-order importance. The point of view adopted by the researcher can be decisive in this kind of analyses. In fact, current research in mathematics education seems to be affected by two main limitations. On the one hand, the "mentalism" vision of mathematics provided by our western culture impedes us to consider the ostensive as a constitutive dimension of the mathematical activity and the mathematical knowledge produced. It is difficult for us to accept ostensive objects as real tools, and thus mathematical objects of full right, and not only "earthly representations" of a conceptual world located in the immateriality of our mind. On the other hand, most of the research focused on the difficulties raised by the use of semiotic objects in the teaching and learning of mathematics tends to acritically assume the epistemological model of mathematics that is dominant in the wise mathematical institution—a model centred in the euclideanism and identifying mathematical knowledge with an organised body of concepts. It seems to forget that this knowledge is the product of complex processes, which have social, cognitive, and cultural components.

The two approaches presented in this paper propose different views of mathematics and mathematics-teaching–learning activities. ATD approach integrates the ostensive dimension of mathematics teaching and learning phenomena in an institutional analysis of large grain. The APC-space frame raises the complementary need to integrate a fine-grained semiotic analysis into a general cognitive model of mathematics teaching-learning processes. For example, the analysis through the APC-space frame has allowed to identify an interesting didactical phenomenon, the so-called *semiotic game* (Arzarello & Paola, 2007; Sabena, 2007). Through the semiotic game, the teacher can use some semiotic resource introduced by the students, e.g. a gesture, to tune with them in the interaction. At the same time, he can introduce the appropriate signs to focus the mathematical object at stake, e.g. words, written inscriptions, etc. In this way, semiotic games constitute an important step in the process of appropriation of the culturally shared meaning of ostensives. They give the students the opportunity of entering in resonance with the teacher's multimodal semiotic resources and through them with the institutional knowledge. APC-space allows to analyze carefully the complex dialectics of the semiotic sets within the semiotic bundle where the teacher develops the semiotic game. But the phenomenon of the semiotic game still needs a didactical-mathematical analysis. For example, an open question regards how to develop a frame within which one can explain why in our examples of semiotic games the so called "Jourdain effect" (Brousseau, 1997) is not likely to appear. To get insights, it may be helpful to insert the APC-space model within epistemological, didactical, and institutional frames, and the ATD appears to provide suitable tools for this purpose.

It is in this sense that, given the above depicted complementarities about their reciprocal needs, with this paper we intended to provide a contribution to the development of suitable frames for "networking theories".

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References

Arzarello, F. (2006). Semiosis as a multimodal process. *Revista Latinoamericana de Investigación en Matemática Educativa*, Special Issue on Semiotics, Culture and Mathematical Thinking, pp. 267–299.

Arzarello, F. (2008). Mathematical landscapes and their inhabitants: perceptions, languages, theories. In M. Niss (Ed.), *Proceedings ICME 10*, Plenary Lecture.

Arzarello, F., Bazzini, L., Ferrara, F., Robutti, O., Sabena, C., & Villa, B. (2006). Will Penelope choose another bridegroom? Looking for an answer through signs, *Proceedings of 30th conference of the international group for the psychology of mathematics education*, Prague.

Arzarello, F., & Olivero, F. (2005). Theories and empirical researches: towards a common framework. In M. Bosch (Ed.), *European research in mathematics education IV, Proceedings of CERME 4*.

Arzarello, F., & Paola, D. (2007). Semiotic games: the role of the teacher. *Proceedings of 31st conference of the international group for the psychology of mathematics education*. 8–15 July 2007: Seoul.

Barquero, B., Bosch, M., & Gascón, J. (2008). Using *research and study courses* for teaching mathematical modelling at university level, *Proceedings of CERME5*.

Bolea P., Bosch. M., Gascón J. (2004). Why is modelling not included in the teaching of algebra at secondary school? *Quaderni di Ricerca in Didattica*, *14*, 125–133.

Bosch, M. (1994). La dimensión ostensiva en la actividad matemática. El caso de la proporcionalidad. Doctoral dissertation, Universitat Autònoma de Barcelona.

Bosch, M., & Chevallard, Y. (1999). La sensibilité de l'activité mathématique aux ostensifs. *Recherches en Didactique des Mathématiques*, *19*(1), 77–124.

Bosch, M., & Gascón, J. (2006). Twenty-five years of the didactic transposition. *ICMI Bulletin*, *58*, 51–63.

Brousseau, G. (1997). Theory of didactical situations in mathematics. Didactique des Mathématiques 1970–1990. Dordrecht: Kluwer Academic Publishers.

Chevallard, Y. (1999). L'analyse de pratiques professorales dans la théorie anthropologique du didactique. *Recherches en Didactique des Mathématiques*, *19*(2), 221–266.

Chevallard, Y. (2002). Organiser l'étude. 3. Écologie & régulation. *Actes de la XI école d'été de didactique* (p. 41–56). Grenoble: La pensée sauvage.

Chevallard, Y. (2004). Vers une didactique de la codisciplinarité. Notes sur une nouvelle épistémologie scolaire. *Journées de didactique comparée*. Lyon.

Chevallard, Y. (2006). Steps towards a new epistemology in mathematics education. In: Bosch, M. (Ed.), *Proceedings of the IV congress of the European society for research in mathematics education (CERME 4)* (p. 1254–1263).

Chevallard, Y., Bosch, M., & Gascón, J. (1997). Estudiar matemáticas. El eslabón perdido entre la enseñanza y el aprendizaje. Barcelona: ICE/Horsori.

Duval R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics*, (*61*), 103–131.

Ernest, P., (2006). A semiotic perspective of mathematical activity: the case of number. *Educational Studies in Mathematics*, (61), 67–101.

Ferrara, F. (2006). Acting and interacting with tools to understand Calculus Concepts, *PhD dissertation*, Turin University.

Gallese, V., & Lakoff, G. (2005). The brain's concepts: The role of the sensory-motor system in conceptual knowledge. *Cognitive Neuropsychology*, *22*, 455–479.

García, J. (2005). La modelización como herramienta de articulación de la matemática escolar. De la proporcionalidad a las herramientas funcionales. Doctoral dissertation. Universidad de Jaén.

García, F. J., Gascón, J., Ruiz Higueras, L., & Bosch, M. (2006). Mathematical modelling as a tool for the connection of school mathematics. *Zentralblatt für Didaktik der Mathematik*, *38*(3), 226–246.

Goldin-Meadow, S. (2003). *Hearing gestures: how our hands help us think*. Cambridge: Harvard University Press.

Lakatos, I. (1978). Mathematics, science and epistemology: philosophical papers, vol 2. Cambridge: University Press.

McNeill, D. (1992). *Hand and mind: what gestures reveal about thought*. Chicago: Chicago University Press.

Nemirovsky, R. (2003). Three conjectures concerning the relationship between body activity and understanding mathematics. In: N. A. Pateman, B. J. Dougherty & J. T. Zilliox (eds.), *Proceeding of PME 27*, (1, pp. 103–135), Honolulu, Hawaii.

Radford, L., Bardini, C., Sabena, C., Diallo, P., & Simbagoye, A. (2005). On embodiment, artifacts, and signs: a semiotic-cultural perspective on mathematical thinking. In H. L. Chick & J. L. Vincent (Eds.), *Proceedings of PME*, *29*(4) 113–120.

Rodríguez, E. (2005). Metacognición, matemáticas y resolución de problemas: una propuesta integradora desde el enfoque antropológico. Doctoral dissertation. Universidad Complutense de Madrid.

Rodríguez, E., Bosch, M., & Gascón, J. (2004). ¿Qué papel se asigna a la resolución de problemas en el currículum actual de matemáticas? In C. Castro y M. Gómez (Eds.), *Análisis del currículo actual de matemáticas y posibles alternativas*, (pp. 95–118). Barcelona: Edebé.

Rodríguez, E., Bosch, M., & Gascón, J. (2008). An anthropological approach to metacognition: praxeologies and study and research courses.

Ruiz, N., Bosch, M., & Gascón, J. (2008). The functional algebraic modelling at Secondary level *Proceedings of CERME5*.

Sabena, C. (2007). Body and signs: a multimodal semiotic approach to teaching–learning processes in early Calculus, *PhD dissertation*, Turin University.

Sáenz-Ludlow, A., & Presmeg, N. (Eds.) 2006, *Special Issue of Educational Studies in Mathematics*, 61, 1–2.

Seitz, J. A. (2000). The bodily basis of thought, new ideas in psychology. *An International Journal of Innovative Theory in Psychology*, *18*(1), 23–40.

Thurston, W. P. (1994). On proof and progress in mathematics. *Bulletin of the American Mathematical Society*, *30/2*, 161–177.

Wilson, M. (2002). Six views of embodied cognition. *Psychonomic Bulletin and Review*, *9*(4), 625–636. (http://www.indiana.edu/~cogdev/labwork/WilsonSixViewsofEmbodiedCog.pdf).